

Free Probability for Pairs of Faces IV: Bi-Free Extremes in the Plane

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ABSTRACT. We compute the bi-free max-convolution which is the operation on bi-variate distribution functions corresponding to the max-operation with respect to the spectral order on bi-free bi-partite two-faced pairs of hermitian non-commutative random variables. With the corresponding definitions of bi-free max-stable and max-infinitely-divisible laws their determination becomes in this way a classical analysis question.

0. Introduction

The definition and classification of free max-stable laws in [2] had been an unexpected addition to the list of free probability analogues to classical probability items. Here we take the first step in a similar direction in bi-free probability [9]. We show that there is a simple formula for computing the bi-free extremal convolution of probability measures in the plane. This corresponds to computing the distribution of $(a \vee c, b \vee d)$, where (a, b) and (c, d) are two bi-free two-faced pairs of commuting hermitian operators. Like the free extremal convolution on \mathbb{R} defined in [2], the bi-free extremal convolution in the plane reduces the question of bi-free max-stable laws in the plane to an analysis problem in the classical context which we won't pursue here.

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The derivation of the formula for the extremal bi-free convolution relies on the partial bi-free R -transform we found in [10]. Obviously our result here complements the recent work on operations on bi-free bi-partite hermitian two-faced pairs ([10], [7], [5], [11], [8]).

The present paper has three more sections besides the introduction. Section one contains preliminaries. Section two derives the main technical result, the bi-free extremal convolution for the distributions of hermitian bi-partite pairs in the case the variables are projections. The third section gives the formula for the extremal bi-free convolution in the general case of hermitian variables and the definitions of bi-free max-stable and max-infinitely-divisible laws.

1. Preliminaries

1.1. Throughout the preliminaries (M, τ) will denote a W^* -probability space, that is M is a von Neumann algebra and τ a normal state. If \mathcal{A} is a C^* -algebra by $\text{Proj}(\mathcal{A})$ we shall denote the hermitian projections $P = P^* = P^2 \in \mathcal{A}$. If $\mathcal{A} = N$ a von Neumann algebra and $P, Q \in \text{Proj}(N)$ by $P \vee Q$ and $P \wedge Q$ we denote the least projection \geq both P and Q and, respectively, the largest projection \leq both P and Q . If \mathcal{H} is the Hilbert space on which N acts then $P \vee Q$ is the orthogonal projection onto $\overline{P\mathcal{H} + Q\mathcal{H}}$, while $P \wedge Q$ is the orthogonal projection onto $P\mathcal{H} \cap Q\mathcal{H}$.

1.2. If $M_h = \{m \in M \mid m = m^*\}$, we recall that the spectral order ([1]; see also [2]) on M_h is defined by $a \prec b$ if the spectral projections satisfy

$$E(a; [t, \infty)) \leq E(b; [t, \infty))$$

for all $t \in \mathbb{R}$. Clearly this extends to self-adjoint operator affiliated with M , since $E(a; [t, \infty)) \leq E(b; [t, \infty))$ makes sense also under this more general assumption. If $a, b \in M_h$ then $a \vee b$ and $a \wedge b$ are defined by

$$E(a \vee b; (t, \infty)) = E(a; (t, \infty)) \vee E(b; (t, \infty))$$

and

$$E(a \wedge b; [t, \infty)) = E(a; [t, \infty)) \wedge E(b; [t, \infty)).$$

These definitions clearly work also more generally for affiliated self-adjoint operators. In essence the operators a, b are replaced by the right-continuous decreasing family of projections $R \ni t \rightarrow E(a, (t, \infty)) \in \text{Proj}(M)$ and, respectively, by the left-continuous decreasing family of projections $\mathbb{R} \ni t \rightarrow E(a; [t, \infty)) \in \text{Proj}(M)$.

1.3. A basic fact underlying free extreme values is the following:

Lemma 1.1. *If $P, Q \in \text{Proj}(M)$ are freely independent in (M, τ) , then*

$$\tau(P \vee Q) = \min(\tau(P) + \tau(Q), 1)$$

and

$$\tau(P \wedge Q) = \max(0, \tau(P) + \tau(Q) - 1).$$

This is a well-known fact. For references see [2] where this is Lemma 2.1 and see [12], [13] for computations. Remark that it is not necessary to require that τ be tracial, since its restrictions to the von Neumann algebra generated by two free hermitian operators is always tracial.

We also recall from [2] the definitions of extremal free convolutions for probability measures on \mathbb{R} . If μ is a probability measures on \mathbb{R} , its distribution function is $F(t) = \mu((-\infty, t])$. If μ and ν are probability measure on \mathbb{R} with distribution functions $F(t)$ and $G(t)$, then $\mu \boxplus \nu$ and $\mu \boxminus \nu$ are defined via their distribution functions $H(t) = \max(0, F(t) + G(t) - 1)$ and, respectively, $K(t) = \min(F(t) + G(t), 1)$. If μ and ν are the distributions of $a, b \in M_h$ with respect to τ , then $\mu \boxplus \nu$ and $\mu \boxminus \nu$ are the distributions of $a \vee b$ and $a \wedge b$. It is also convenient to have corresponding operations on distribution functions of probability measures on \mathbb{R} . If F, G are two such distribution functions, then $F \boxplus G = (F + G - 1)_+$ and $F \boxminus G = \min(F + G, 1)$.

1.4. We conclude the section of preliminaries by recalling some basics about the free R -transform and the partial bi-free R -transform.

If $a \in (\mathcal{A}, \varphi)$ is a non-commutative random-variable in a Banach-algebra probability space, let $G_a(z) = \varphi((z1 - a)^{-1})$ be the Green-function, or Cauchy-transform of the distribution of a , which is a holomorphic function in a neighborhood of $\infty \in \mathbb{C} \cup \{\infty\}$ on the Riemann sphere. Then $K_a(z)$ defined in a neighborhood of $0 \in \mathbb{C}$ and taking values in $\mathbb{C} \cup \{\infty\}$ is the inverse of G_a , that is $K_a(0) = \infty$ and $G_a(K_a(z)) = z$. Further $R_a(z) = K_a(z) - z^{-1}$ is defined in a neighborhood of 0 . If a and b are free, then $R_{a+b}(z) = R_a(z) + R_b(z)$ in a neighborhood of 0 .

If (a, b) is a two-faced bi-free pair in (\mathcal{A}, φ) we consider

$$G_{a,b}(z, w) = \varphi((z1 - a)^{-1}(w1 - b)^{-1})$$

defined in a neighborhood of $(\infty, \infty) \in (\mathbb{C} \cup \{\infty\})^2$. The reduced partial bi-free R -transform of (a, b) is

$$\tilde{\mathcal{R}}_{a,b}(z, w) = 1 - \frac{zw}{G_{a,b}(K_a(z), K_b(w))}$$

defined in a neighborhood of $(0, 0) \in \mathbb{C}^2$. If (a, b) and (c, d) are bi-free in (\mathcal{A}, φ) , then

$$\tilde{\mathcal{R}}_{a+c, b+d}(z, w) = \tilde{\mathcal{R}}_{a,b}(z, w) + \tilde{\mathcal{R}}_{c,d}(z, w)$$

in a neighborhood of $(0, 0) \in \mathbb{C}^2$ (see [10]).

If (a, b) are commuting hermitian operators in a C^* -probability space (\mathcal{A}, φ) , then the joint distribution of (a, b) is given by a probability measure on \mathbb{R}^2 with compact support $\mu_{a,b}$ and

$$G_{a,b}(z, w) = \iint (z - x)^{-1} (w - y)^{-1} d\mu_{a,b}(x, y).$$

The measures $\mu_{a,b}$, $\mu_{c,d}$ and $\mu_{a+c, b+d}$ when (a, b) and (c, d) are bi-free in (\mathcal{A}, φ) are related by additive bi-free convolution

$$\mu_{a,b} \boxplus \boxplus \mu_{c,d} = \mu_{a+c, b+d}.$$

2. Two-faced pairs of commuting projections

In this section we shall compute the bi-free extremal convolution in the case of the distributions of commuting projections. This is the bi-free generalization of the free probability result in Lemma 1.1. We begin with a sequence of lemmas.

Lemma 2.1. *Let (\mathcal{A}, φ) be a C^* -probability space and let $P = P^* = P^2 \in \mathcal{A}$ and $\varphi(P) = p$. Then we have*

$$G_P(z) = \frac{p}{z-1} + \frac{1-p}{z} = \frac{z+p-1}{z(z-1)}$$

and

$$zK_P(K_P - 1) = K_P + p - 1.$$

Lemma 2.2. *Let (\mathcal{A}, φ) be a C^* -probability space and let (P, Q) be a two-faced pair in \mathcal{A} , so that $P = P^*$, $Q = Q^*$, $P = P^2$, $Q = Q^2$, $[P, Q] = 0$, $\varphi(P) = p$, $\varphi(Q) = q$, $\varphi(PQ) = r$. Then we have:*

$$G_{P,Q}(z, w) = \frac{(z+p-1)(w+q-1) + (r-pq)}{zw(z-1)(w-1)}.$$

Lemma 2.3. *Under the same assumptions as in Lemma 2.2, we have:*

$$G_{P,Q}(K_P, K_Q) = \frac{zw((K_P + p - 1)(K_Q + q - 1) + (r - pq))}{(K_P + p - 1)(K_Q + q - 1)}.$$

Lemma 2.4. *Under the same assumptions as in Lemma 2.2, we have:*

$$\tilde{R}_{P,Q}(z, w) = \frac{r - pq}{(K_P(z) + p - 1)(K_Q(w) + q - 1) + r - pq}.$$

The proofs of the preceding four lemmata are straightforward computations and will be omitted.

Lemma 2.5. *Let μ be a probability measure on $[0, 2]^2 \subset \mathbb{R}^2$ and let*

$$G(z, w) = \iint (z - x)^{-1}(w - y)^{-1} d\mu(x, y).$$

Let further $z_n \in (2, \infty)$, $w_n \in (2, \infty)$ be such that $z_n \rightarrow 2$, $w_n \rightarrow 2$ as $n \rightarrow \infty$. Then we have:

$$\lim_{n \rightarrow \infty} (z_n - 2)(w_n - 2)G(z_n, w_n) = \mu(\{(2, 2)\}).$$

Proof. Let $F_n(x, y) = (z_n - 2)(z_n - x)^{-1}(w_n - 2)(w_n - y)^{-1}$ where $(x, y) \in [0, 2]^2$. Then $|F_n| \leq 1$ for $(x, y) \in [0, 2]^2$ and F_n converges pointwise to the indicator function of $\{(2, 2)\}$. By Lebesgue's dominated convergence theorem we have

$$\lim_{n \rightarrow \infty} \iint F_n(x, y) d\mu(x, y) = \mu(\{(2, 2)\})$$

which is what we wanted to prove. \square

Lemma 2.6. *In a C^* -probability space (\mathcal{A}, φ) let (P, Q) and (P', Q') be bi-free two-faced pairs so that $P = P^* = P^2$, $Q = Q^* = Q^2$, $P' = P'^* = P'^2$, $Q' = Q'^* = Q'^2$, $[P, Q] = [P', Q'] = [P, Q'] = [P', Q] = 0$ and $\varphi(P) = p$, $\varphi(Q) = q$, $\varphi(PQ) = r$, $\varphi(P') = p'$, $\varphi(Q') = q'$, $\varphi(P'Q') = r'$ and let $\delta = r - pq$, $\delta' = r' - p'q'$. Then for (z, w) in some neighborhood of $(0, 0) \in \mathbb{C}^2$ we have*

$$\begin{aligned} G_{P+P', Q+Q'}(K_{P+P'}(z), K_{Q+Q'}(w)) &= \\ &= zw(1 - (1 + \delta^{-1}(K_P(z) + p - 1)(K_Q(w) + q - 1))^{-1} \\ &\quad - (1 + \delta'^{-1}(K_{P'}(z) + p' - 1)(K_{Q'}(w) + q' - 1))^{-1})^{-1}. \end{aligned}$$

In case $\delta = 0$ we set here $(1 + \delta^{-1}(K_P(z) + p - 1)(K_Q(w) + q - 1))^{-1} = 0$ and we adopt also a similar rule if $\delta' = 0$.

Proof. We have

$$\begin{aligned} G_{P+P', Q+Q'}(K_{P+P'}(z), K_{Q+Q'}(w)) &= zw(1 - \tilde{\mathcal{R}}_{P+P', Q+Q'}(z, w))^{-1} \\ &= zw(1 - \tilde{\mathcal{R}}_{P, Q}(z, w) \\ &\quad - \tilde{\mathcal{R}}_{P', Q'}(z, w))^{-1}. \end{aligned}$$

Note that if $\delta = 0$, P and Q are classically independent, so that $\tilde{\mathcal{R}}_{P, Q}(z, w) = 0$ and similarly $\tilde{\mathcal{R}}_{P', Q'}(z, w) = 0$ if $\delta' = 0$. On the other hand, Lemma 2.4 gives that $\tilde{\mathcal{R}}_{P, Q}(z, w) = (1 + \delta^{-1}(K_P(z) + p - 1)(K_Q(w) + q - 1))^{-1}$ if $\delta \neq 0$ and a similar fact for $\tilde{\mathcal{R}}_{P', Q'}(z, w)$. \square

Lemma 2.7. *Under the same assumptions as in Lemma 2.6 we have for (z, w) in some neighborhood of $(0, 0) \in \mathbb{C}^2$ that*

$$\begin{aligned} &(K_{P+P'}(z) - 2)(K_{Q+Q'}(w) - 2)G_{P+P', Q+Q'}(K_{P+P'}(z), K_{Q+Q'}(w)) \\ &= (K_P(z) + K_{P'}(z) - z^{-1} - 2)(K_Q(w) + K_{Q'}(w) - w^{-1} - 2) \\ &\quad zw(1 - (1 + \delta^{-1}(K_P(z) + p - 1)(K_Q(w) + q - 1))^{-1} \\ &\quad - (1 + \delta'^{-1}(K_{P'}(z) + p' - 1)(K_{Q'}(w) + q' - 1))^{-1})^{-1}, \end{aligned}$$

this being an equality of holomorphic functions.

Proof. This follows from the preceding lemma after multiplication with

$$\begin{aligned} &(K_{P+P'}(z) - 2)(K_{Q+Q'}(w) - 2) = \\ &(K_P(z) + K_{P'}(z) - z^{-1} - 2)(K_Q(w) + K_{Q'}(w) - w^{-1} - 2). \end{aligned}$$

Since the conclusion of Lemma 2.6 was actually an equality of germs of holomorphic functions near $(0, 0) \in \mathbb{C}^2$ it might seem that there may be a problem with infinities when z or w is 0. It is easily seen looking at the right-hand side that this is not the case since $(K_P(z) + K_{P'}(z) - z^{-1} - 2)z$ is holomorphic in a neighborhood of $z = 0$ and $(K_Q(w) + K_{Q'}(w) - w^{-1} - 2)w$ is holomorphic in a neighborhood of $w = 0$, while

$$\begin{aligned} &(1 - (1 + \delta^{-1}(K_P(z) + p - 1)(K_Q(w) + q - 1))^{-1} \\ &\quad - (1 + \delta'^{-1}(K_{P'}(z) + p' - 1)(K_{Q'}(w) + q' - 1))^{-1}) \end{aligned}$$

is holomorphic in a neighborhood of $(0, 0) \in \mathbb{C}^2$ and $\neq 0$. \square

Lemma 2.8. *Assuming $p > 0$, $q > 0$, $p' > 0$, $q' > 0$, the equality which is the conclusion of Lemma 2.7 for (z, w) in a neighborhood of $(0, 0) \in \mathbb{C}^2$ extends analytically to (z, w) in a neighborhood of $[0, \infty)^2 \subset \mathbb{C}^2$.*

Proof. Let t_0 and s_0 be the least upper bounds of the supports of the probability measures $\mu_{P+P'}$ and $\mu_{Q+Q'}$ on \mathbb{R} . Then $G_{P+P'}(z)$ on $(t_0, \infty]$ and $G_{Q+Q'}(w)$ on $(s_0, \infty]$ are strictly decreasing taking the values $[0, \infty)$ so that $K_{P+P'}(z)$ and $K_{Q+Q'}(w)$ have analytic continuations along $[0, \infty)$ to a neighborhood of $[0, \infty)$ (the functions are viewed as taking values in the Riemann sphere $\mathbb{C} \cup \{\infty\}$). This implies the analytic extension of

$$\begin{aligned} & zw(K_P(z) + K_{P'}(z) - z^{-1} - 2)(K_Q(w) + K_{Q'}(w) - w^{-1} - 2) \\ &= zw(K_{P+P'}(z) - 2)(K_{Q+Q'}(w) - 2) \end{aligned}$$

as a holomorphic function, taking values in \mathbb{C} , to a neighborhood of $[0, \infty)^2 \subset \mathbb{C}^2$. On the other hand, similarly, since $p > 0$, $p' > 0$, $q > 0$, $q' > 0$ the functions $K_P(z)$, $K_{P'}(z)$, $K_Q(w)$, $K_{Q'}(w)$ have analytic continuations to a neighborhood of $[0, \infty)$ taking values for $z, w \in [0, \infty)$ in $(1, \infty]$. If $z, w \in [0, \infty)$, $(1 + \delta^{-1}(K_P(z) + p - 1)(K_Q(w) + q - 1))^{-1}$ is well-defined when $\delta \neq 0$ since $(K_P(z) + p - 1)(K_Q(w) + q - 1) > pq > 0$ while either $\delta^{-1} > 0$ or $0 > \delta^{-1} \geq -p^{-1}q^{-1}$ so that the quantity to be inverted is > 0 . Similar reasoning takes care of the term involving δ' . From $[0, \infty)^2$ the extension goes over to a neighborhood.

For the analytic extension of

$$G_{P+P', Q+Q'}(K_{P+P'}(z), K_{Q+Q'}(w))$$

it suffices to remark that $G_{P+P', Q+Q'}(z, w)$ is analytic in a neighborhood of $(t_0, \infty] \times (s_0, \infty] \subset (\mathbb{C} \cup \{\infty\})^2$ and $K_{P+P'}(z)$, $K_{Q+Q'}(w)$ on $[0, \infty)$ take values in $(t_0, \infty]$ and $(s_0, \infty]$, respectively. That

$$\begin{aligned} & (1 - (1 + \delta^{-1}(K_P(z) + p - 1)(K_Q(w) + q - 1))^{-1} \\ & \quad - (1 + \delta'^{-1}(K_{P'}(z) + p' - 1)(K_{Q'}(w) + q' - 1))^{-1}) \end{aligned}$$

is $\neq 0$ for $(z, w) \in [0, \infty)^2$ follows from the fact the left-hand side is finite and

$$\begin{aligned} & zw(K_P(z) + K_{P'}(z) - z^{-1} - 2)(K_Q(w) + K_{Q'}(w) - w^{-1} - 2) \\ &= zw(K_{P+P'}(z) - 2)(K_{Q+Q'}(w) - 2) \neq 0. \end{aligned}$$

□

Lemma 2.9. *If (\mathcal{A}, φ) is a W^* -probability space and $P = P^* = P^2 \in \mathcal{A}$, $P' = P'^* = P'^2 \in \mathcal{A}$ then $P \wedge P' = E(P + P', \{2\})$.*

The preceding lemma is a well-known fact.

Lemma 2.10. *Let (\mathcal{A}, φ) be a W^* -probability space. Then under the assumptions of Lemma 2.6, if $p + p' - 1 > 0$ and $q + q' - 1 > 0$ we have*

$$\begin{aligned} & \varphi((P \wedge P')(Q \wedge Q')) \\ &= (p + p' - 1)(q + q' - 1)(1 - (1 + \delta^{-1}pq)^{-1} - (1 + \delta'^{-1}p'q')^{-1})^{-1}. \end{aligned}$$

Here in case $\delta = 0$ we set $(1 + \delta^{-1}pq)^{-1} = 0$ and adopt a similar rule if $\delta' = 0$. In case $p + p' - 1 \leq 0$ or $q + q' - 1 \leq 0$ we have $\varphi((P \wedge P')(Q \wedge Q')) = 0$.

Proof. As recorded in Lemma 1.1, we have

$$\varphi(P \wedge P') = (p + p' - 1)_+, \quad \varphi(Q \wedge Q') = (q + q' - 1)_+$$

after observing that P and P' being free, the restriction of φ to the algebra generated by P and P' is a trace and a similar fact for Q and Q' . Clearly, if $\varphi(P \wedge P') = 0$ or $\varphi(Q \wedge Q') = 0$, we must have also $\varphi((P \wedge P')(Q \wedge Q')) = 0$ since $[P \wedge P', Q \wedge Q'] = 0$ and $0 \leq (P \wedge P')(Q \wedge Q') \leq P \wedge P'$. Thus we are left with proving the lemma when $p + p' - 1 > 0$ and $q + q' - 1 > 0$.

Thus $\mu_{P+P'}(\{2\}) = p + p' - 1 > 0$, $\mu_{Q+Q'}(\{2\}) = q + q' - 1 > 0$ and $\text{supp } \mu_{P+P'} \subset [0, 2]$, $\text{supp } \mu_{Q+Q'} \subset [0, 2]$. By considerations along the lines in the proof of Lemma 2.8, if $t_n \in (0, \infty)$, $t_n \uparrow \infty$ then $K_P(t_n) \downarrow 1$, $K_{P'}(t_n) \downarrow 1$, $K_Q(t_n) \downarrow 1$, $K_{Q'}(t_n) \downarrow 1$, $K_{P+P'}(t_n) \downarrow 2$, $K_{Q+Q'}(t_n) \downarrow 2$. Taking $z = w = t_n$ in the equality in Lemma 2.7 extended according to Lemma 2.8, we get that the limit of the left-hand side in view of Lemma 2.5 is

$$\begin{aligned} \mu_{P+P', Q+Q'}(\{(2, 2)\}) &= \varphi(E(P + P', \{2\})E(Q + Q', \{2\})) \\ &= \varphi((P \wedge P')(Q \wedge Q')). \end{aligned}$$

On the other hand,

$$\begin{aligned} & (K_P(t_n) + K_{P'}(t_n) - t_n^{-1} - 2)t_n \\ &= (K_P(t_n) - 1)t_n + (K_{P'}(t_n) - 1)t_n - 1 \\ &= G_P(K_P(t_n))(K_P(t_n) - 1) + G_{P'}(K_{P'}(t_n))(K_{P'}(t_n) - 1) - 1 \end{aligned}$$

and this converges as $n \rightarrow \infty$, by the simpler analogue of Lemma 2.5 for Cauchy transforms in one variable, to $p + p' - 1$. Similarly $(K_Q(t_n) + K_{Q'}(t_n) - t_n^{-1} - 2)t_n$ converges to $q + q' - 1$. On the other hand

$$\begin{aligned} & (1 - (1 + \delta^{-1}(K_P(t_n) + p - 1)(K_Q(t_n) + q - 1))^{-1} \\ & \quad - (1 + \delta'^{-1}(K_{P'}(t_n) + p' - 1)(K_{Q'}(t_n) + q' - 1))^{-1}) \end{aligned}$$

converges to $(1 - (1 + \delta^{-1}pq)^{-1} - (1 + \delta'^{-1}p'q')^{-1})$ no matter whether δ and δ' are $\neq 0$ or $= 0$. \square

The last lemma after some simple algebraic work on the formulae will give the final result of the computations in this section, which we record as a theorem.

Theorem 2.1. *Let (\mathcal{A}, φ) be a W^* -probability space and let $P = P^* = P^2 \in \mathcal{A}$, $Q = Q^* = Q^2 \in \mathcal{A}$, $P' = P'^* = P'^2 \in \mathcal{A}$, $Q' = Q'^* = Q'^2 \in \mathcal{A}$ be such that $[P, Q] = 0$, $[P', Q] = [P, Q'] = 0$, $[P', Q'] = 0$ and (P, Q) and (P', Q') are bi-free in (\mathcal{A}, φ) . Then we have $\varphi(P \wedge P') = (\varphi(P) + \varphi(P') - 1)_+$, $\varphi(Q \wedge Q') = (\varphi(Q) + \varphi(Q') - 1)_+$ and if $\varphi(P \wedge P') > 0$, $\varphi(Q \wedge Q') > 0$, $\varphi(PQ) > 0$, $\varphi(P'Q') > 0$ then we have*

$$\frac{\varphi(P \wedge P')\varphi(Q \wedge Q')}{\varphi((P \wedge P')(Q \wedge Q'))} = \frac{\varphi(P)\varphi(Q)}{\varphi(PQ)} + \frac{\varphi(P')\varphi(Q')}{\varphi(P'Q')} - 1.$$

If any of the numbers $\varphi(P \wedge P')$, $\varphi(Q \wedge Q')$, $\varphi(PQ)$, $\varphi(P'Q')$ is 0, then $\varphi((P \wedge P')(Q \wedge Q')) = 0$.

Proof. The formulae for $\varphi(P \wedge P')$, $\varphi(Q \wedge Q')$ are not new (see Lemma 1.1) and it is obvious that if any of $\varphi(P \wedge P')$, $\varphi(Q \wedge Q')$, $\varphi(PQ)$, $\varphi(P'Q')$ is 0, then so is $\varphi((P \wedge P')(Q \wedge Q'))$. Thus using the notation $\varphi(P) = p$, $\varphi(P') = p'$, $\varphi(Q) = q$, $\varphi(Q') = q'$, $\varphi(PQ) = r$, $\varphi(P'Q') = r'$ which we used in the lemmata, we may assume $p + p' > 1$, $q + q' > 1$, $r > 0$, $r' > 0$. Turning to the result of Lemma 2.10, remark that if $\delta = r - pq \neq 0$ then

$$(1 + \delta^{-1}pq)^{-1} = \delta(\delta + pq)^{-1} = (r - pq)r^{-1} = 1 - pqr^{-1}.$$

If $\delta = 0$, then $r = pq$ and $1 - pqr^{-1} = 0$ which is in agreement with the rule that $(1 + \delta^{-1}pq)^{-1}$ is 0 if $\delta = 0$. A similar remark about δ' . Hence the right-hand side of the formula in Lemma 2.10 is

$$\varphi(P \wedge P')\varphi(Q \wedge Q')(pqr^{-1} + p'q'r'^{-1} - 1)^{-1}$$

so that the formula gives

$$\begin{aligned} \frac{\varphi(P \wedge P')\varphi(Q \wedge Q')}{\varphi((P \wedge P')(Q \wedge Q'))} &= pqr^{-1} + p'q'r'^{-1} - 1 \\ &= \frac{\varphi(P)\varphi(Q)}{\varphi(PQ)} + \frac{\varphi(P')\varphi(Q')}{\varphi(P'Q')} - 1. \end{aligned}$$

□

3. Bi-free max-convolution in the plane

In this section (\mathcal{A}, φ) will be a von Neumann algebra with a normal state φ . If (a, b) is a bi-partite hermitian two-faced pair in (\mathcal{A}, φ) , that is a pair of commuting hermitian operators $a, b \in \mathcal{A}$, let $E(a, b; \omega)$ denote its spectral measure where $\omega \subset \mathbb{R}^2$ is a Borel set and let

$\mu_{a,b}(\omega) = \varphi(E(a, b; \omega))$ be the probability measure on \mathbb{R}^2 which is the distribution of (a, b) . Let further $F_{a,b}(s, t) = \mu_{a,b}((-\infty, s] \times (-\infty, t])$ be the distribution function of the measure $\mu_{a,b}$. We recall that such functions $F(s, t)$ are such that $s_1 \leq s_2, t_1 \leq t_2 \Rightarrow F(s_1, t_1) \leq F(s_2, t_2)$, $s_n \downarrow s_0, t_n \downarrow t_0 \Rightarrow F(s_n, t_n) \downarrow F(s_0, t_0)$ as $n \rightarrow \infty$ and $s_1 \leq s_2, t_1 \leq t_2 \Rightarrow F(s_2, t_2) - F(s_1, t_2) - F(s_2, t_1) + F(s_1, t_1) \geq 0$. Moreover, since this is the distribution function of a probability measure with compact support, we have $0 \leq F(s, t) \leq 1$ and there is $C > 0$ so that $F(s, t) = 0$ if $\min(s, t) \leq -C$ and $F(s, t) = 1$ if $\min(s, t) \geq C$. If we want to deal with probability measures without a condition of compact support, we will require that F be defined on $[-\infty, \infty)^2$ and satisfy $F(s, t) = 0$ if $\min(s, t) = -\infty$ and $\lim_{n \uparrow +\infty} F(n, n) = 1$.

If (a, b) and (a', b') are bi-free bi-partite hermitian pairs, it's always possible to find a realization in a von Neumann algebra (\mathcal{A}, φ) of the joint distribution so that $[a, b'] = [a', b] = 0$. Note further that the joint distribution of $(a \vee a', b \vee b')$ does not depend on the realization, but only on the distributions $\mu_{a,b}, \mu_{a',b'}$ since

$$\begin{aligned} & E(a \vee a', b \vee b'; (-\infty, s] \times (-\infty, t]) \\ &= E(a \vee a'; (-\infty, s]) E(b \vee b'; (-\infty, t]) \\ &= (E(a; (-\infty, s]) \wedge E(a'; (-\infty, s])) (E(b; (-\infty, t]) \wedge E(b'; (-\infty, t])) \end{aligned}$$

and $\varphi(E(a \vee a', b \vee b'; (-\infty, s] \times (-\infty, t]))$ can be computed using the results in Section 2 from the distributions of the bi-free two-faced pairs $(E(a; (-\infty, s]), E(b; (-\infty, t]))$ and $(E(a'; (-\infty, s]), E(b'; (-\infty, t]))$.

Definition 3.1. If F and G are distribution functions of probability measures with compact support on \mathbb{R}^2 we define their bi-free max-convolution or alternatively also called bi-free upper extremal convolution $H = F \boxed{\vee \vee} G$ to be such that if F_j, G_j, H_j ($j = 1, 2$) are their distribution function marginals we have $H_j = F_j \boxed{\vee} G_j$ ($j = 1, 2$) and

$$\frac{H_1(s)H_2(t)}{H(s, t)} = \frac{F_1(s)F_2(t)}{F(s, t)} + \frac{G_1(s)G_2(t)}{G(s, t)} - 1$$

if $F(s, t) > 0, G(s, t) > 0, H_1(s) > 0, H_2(t) > 0$ and $H(s, t) = 0$ otherwise.

That the above gives a well-defined distribution function of a probability measure with compact support is a consequence of the discussion preceding the definition and of Theorem 2.1. Note also that to see that the distribution function of $a \wedge a'$ is given by $F_a \boxed{\vee} F_{a'}$ it is not necessary to assume a and a' are in a tracial W^* -probability space, since the restriction of φ to the weak closure of the $*$ -algebra generated

by $\{I, a, a'\}$ will be a tracial normal state. In essence, $\boxed{\vee}\boxed{\vee}$ gives the distribution of $(a \vee a', b \vee b')$ in the realizations of the joint distributions of (a, a') and (b, b') where the commutations $[a, b'] = [a', b] = 0$ hold.

The further remark is that actually a, b, a', b' only appear here via their spectral scales $E(a; (-\infty, t])$ etc. and thus *the operations extend to affiliated unbounded self-adjoint operators and distributions of any probability measures on \mathbb{R}^2 .*

Having defined $\boxed{\vee}\boxed{\vee}$ we can now define bi-free max-stable and bi-free max-infinitely divisible laws on \mathbb{R}^2 .

Definition 3.2. A distribution function F of a probability measure on \mathbb{R}^2 is bi-freely max-stable if there are $a_n, b_n, c_n, d_n \in \mathbb{R}$, $a_n > 0$, $c_n > 0$ so that

$$\underbrace{(F \boxed{\vee}\boxed{\vee} F \boxed{\vee}\boxed{\vee} F \boxed{\vee}\boxed{\vee} \dots F)}_{n\text{-fold}}(a_n x + b_n, c_n y + d_n) \rightarrow F(x, y)$$

as $n \rightarrow \infty$.

Definition 3.3. A distribution function F of a probability measure on \mathbb{R}^2 is bi-freely max-infinitely-divisible if for each $n \in \mathbb{N}$ there is a distribution function F_n so that

$$\underbrace{F_n \boxed{\vee}\boxed{\vee} F_n \boxed{\vee}\boxed{\vee} \dots F_n}_{n\text{-fold}} = F.$$

Remark 3.1. The definitions in this section show that, in the simplest bi-free case of bi-partite hermitian two-faced pairs that have distributions given by probability measures on \mathbb{R}^2 , the basic extreme value questions about bi-free max-stable and bi-free max-infinitely are transformed by the operation $\boxed{\vee}\boxed{\vee}$ into “classical” questions. Clearly these questions are more difficult than univariate free extreme value questions ([2], [3]). It is a natural question whether like in [2], where free max-stable laws were related to classical “peaks over thresholds”, the “classical” questions to which bi-free extremes laws lead in this simplest case are also related to some classical extremes questions ([4], [6]).

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